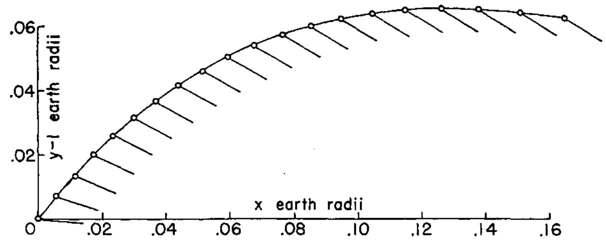

Fig. 2 Trajectory and thrust directions

To give an example, assume a circular orbit of radius $R = 1.075699$ earth radii, corresponding to an altitude of 300 miles. For this orbit $U^2 + V^2 = R^{-1}$ and $\dot{U}^2 + \dot{V}^2 = R^{-4}$. Assume launching conditions $V_0 = 0.585$ and $\theta = 0.928$ rad. Also assume $c = 10{,}000$ fps and $\dot{M} = 0.0036\ \text{sec}^{-1}$. Use initial estimates $T = 0.29$, $l = -0.22$, $m = -30$, $n = 19$, and $B = 0.154$ rad. This input gave convergence in five iterations to the six-figure results $T = 0.289869$, $l = -0.0751076$, $m = -51.8646$, $n = 32.9787$, and $B = 0.153882$. Figure 2 shows the trajectory and thrust directions for equal time intervals. An approximate body of knowledge about possible combinations of V_1, V_2, $\dot{M}$, T, l, m, n, and B must be built up in order to choose reasonable estimated input to the iteration of Eq. (19). To do this, solve Eqs. (1, 2, and 8) with g_1 and g_2 linearized to obtain

$$x = \int_0^t a \cos p \sin(t - w)dw + V_1 \sin t$$

$$2^{1/2}y = \int_0^t a \sin p \sinh 2^{1/2}(t - w)dw + V_2 \sinh 2^{1/2}t - 2^{-1/2}\cosh 2^{1/2}t + 3/2^{1/2} \tag{20}$$

$$\tan p = (l \cosh 2^{1/2}t - 2^{-1/2}n \sinh 2^{1/2}t)/(\cos t - m \sin t)$$

Using Eqs. (20) and (3), it is easy to solve the Newton equations:

$$(X - x)_T = \Delta V_1 \sin T - Y(T)dB + \Delta c \int_0^T a \cos p \sin(T - w)\ \frac{dw}{c}$$

$$(U - u)_T = \Delta V_1 \cos T - V(T)dB + \Delta c \int_0^T a \cos p \cos(T - w)\ \frac{dw}{c}$$

$$2^{1/2}(Y - y)_T = \Delta V_2 \sinh 2^{1/2}T + 2^{1/2}X(T)dB + \Delta c \int_0^T a \sin p \sinh 2^{1/2}(T - w)\ \frac{dw}{c} \tag{21}$$

$$(V - v)_T = \Delta V_2 \cosh 2^{1/2}T + U(T)\, dB + \Delta c \int_0^T a \sin p \cosh 2^{1/2}(T - w)\ \frac{dw}{c}$$

$$10{,}000 \ln[1 - (\dot{M} + \Delta\dot{M})T] = (c + \Delta c)\ln(1 - \dot{M}T)$$

for V_1, V_2, $\dot{M}$, and B for assumed sets of T, l, m, and n. The last of Eqs. (21), which swaps ΔM for Δc by holding the integral of a invariant, was appended to simplify computations.

The methods presented here can be used also when the additional constraint of fixing the angle B is imposed.

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[2] Weinstock, R., *Calculus of Variations* (McGraw-Hill Book Co., Inc., New York, 1952), pp. 57–60.

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Comments

Comment on "Roll Damping of a Fleet Ballistic-Missile Submarine"

John D. Brooks*
U. S. Naval Ordnance Test Station, Pasadena, Calif.

A METHOD has been proposed[1] for estimating the roll-damping moment due to the sail of a fleet ballistic-missile submarine. In the derivation of this method, two-dimensional airfoil theory is used, although the submarine sail is essentially a low-aspect-ratio hydrofoil. As a result, it is believed that the method of Ref. 1 gives results that are greatly in error.

The effect of aspect ratio on roll-damping coefficient (illustrated in Fig. 1) is based on experimental data[2] for untapered, unswept wings. There the coefficient was defined as

$$C_{l_p} = \frac{\partial C_l}{\partial(pb/2V)}$$

since the damping force was found to be linear with roll rate. C_l is the roll-moment coefficient, b the span, p the rolling angular velocity in radians per second, and V the forward velocity. If it is assumed that the submarine-hull diameter is included in the calculation of aspect ratio, the aspect ratio of the submarine used as a numerical example in Ref. 1 is 1.41. From Fig. 1 it is seen that the corresponding C_{l_p} is about -0.09, whereas for the largest value of aspect ratio tested it is -0.37. Thus, assuming steady roll rate, the roll-damping moment in the example is too large by a factor of 4, at least.

Of course, because of the oscillatory motion of the submarine, the full steady-state lift is not developed. However, for very low aspect ratios the lift appears very rapidly when the hydrofoil is given a sudden change in angle of attack. This is illustrated[3] by Fig. 2, which represents the fraction of steady-state lift attained as a function of the distance in chords which the hydrofoil has moved after a sudden change in angle of attack. Reference 3 indicates that, in the limiting case, aspect ratio zero, the steady-state lift is attained immediately. Interpolating, for aspect ratio 1.41, over 75% of the steady lift appears immediately. Since the roll-damping

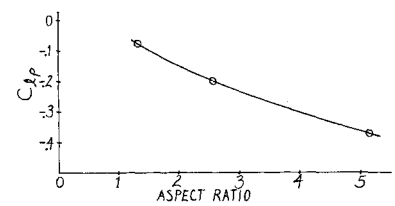

Fig. 1 Variation of roll-damping coefficient with aspect ratio

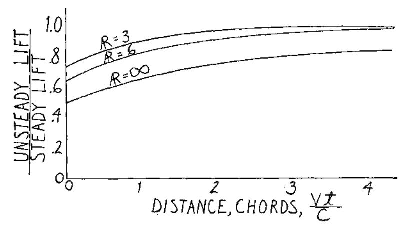

Fig. 2 Unsteady-lift function

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* Physicist, Underwater Ordnance Department.

force is essentially the lift generated by the local apparent angle of attack due to roll rate, Fig. 2 shows that, for the case of the submarine sail, the lift and hence the roll damping should vary with aspect ratio approximately in the same way as in steady-state flow. Consequently, the use of two-dimensional airfoil theory is, as just pointed out, greatly in error, and it would seem that the close agreement between the experimental and theoretical data of Ref. 1 is fortuitous. Perhaps there is a large error in the data used for the calculation, or perhaps other fins or diving planes that contribute to the roll damping were not taken into account.

References

[1] Regan, F. J., "Roll damping of a fleet ballistic-missile submarine," J. Aerospace Sci. **29**, 230 (1962).

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Extension of an Optimum Transfer Note by H. Munick

JOHN D. MULHOLLAND*

University of Cincinnati, Cincinnati, Ohio

Nomenclature

C_1 = circular velocity at r_1
e = eccentricity of outer orbit
G = gravitational constant
M = total two-body mass
p = semilatus rectum of outer orbit
r_1 = transfer departure radius
r_2 = transfer terminal radius
r_p = pericentron distance of outer orbit
x_f = nondimensional circumferential velocity component at r_2 after second thrust impulse
X_p = nondimensional velocity at r_p
λ = nondimensional characteristic velocity $\Delta V/C_1$

It is shown that the recent paper by Munick[1] has more general validity than that which is claimed for it. The result is that the problem of absolute minimum-energy two-impulse transfer between a circular orbit and an external, coplanar, nonintersecting orbit is solved completely. Munick's interpretation of the solution is incorrect, however, in that no absolute minimum exists for $e \geq 1$.

Munick discusses the question of optimum transfer to hyperbolic orbits. The present note is essentially a commentary on that paper, serving to extend its generality. Few equations will be required other than those appearing in Ref. 1; therefore, the equation numbers here will begin with (25),† so that concise and unambiguous reference may be made to Munick's derivation without unnecessary repetition of his equations.

The problem to be considered is that of minimum-energy two-impulse transfer between a circular orbit and an external, coplanar but nonintersecting orbit of unspecified eccentricity. The hyperbolic case is considered by Munick[1] and the circular and elliptic cases by Horner.[2] In all cases, Eq. (1) is the function to be minimized, the nondimensionalized characteristic velocity. The hypotheses assure that the conditions (2 and 4–6) are satisfied. Equation (7), which represents the conservation of energy and angular momentum in the outer orbit, is valid for all possible two-body trajectories. It might be added parenthetically that $\rho = r_1/r_2$ appears in this relation only because the velocities are nondimensionalized by means of

$$C_1 = (GM/r_1)^{1/2} \tag{25}$$

Since the formal expression for λ is not dependent on the type of conic which defines the outer orbit, it is certainly true that the relations (8–13) are valid without need to specify the nature of this orbit. As in Ref. 1, the solution of the minimization problem leads one to consider the expression

$$2X_p b^3 - X_p^2 b^2 - 2b^2 + 1 \tag{26}$$

The quantities X_p and b are related, however, by Eq. (10), which is repeated here for convenience:

$$b = r_1/r_p X_p$$

In the present nondimensional variables, the conservation of angular momentum (area integral) has the form

$$X_p^2 = (r_1/r_p)(p/r_p) = (r_1/r_p)(1 + e) \tag{27}$$

or

$$X_p = b\,(1 + e)$$

which is Eq. (14) and is valid for all e. Thus, for all possible outer orbits, (26) can be written in the form of Eq. (16):

$$[b^2(1 + e) - 1][b^2(1 - e) - 1]$$

Horner[2] has shown that contradictions arise if this quantity is zero, corresponding to an extremal value of λ somewhere other than at the apsides of the outer orbit. He also demonstrated that transfer to the pericentron cannot minimize λ. Thus, λ must be monotonic increasing in x_f, which requires that (16) be positive. As noted by Munick,[1] the first factor of (16) can be written as

$$b^2(1 + e) - 1 = (r_1/r_p) - 1 < 0 \tag{28}$$

which is negative due to the initial hypotheses. On the other hand, since e is always non-negative,

$$b^2(1 - e) - 1 = [b^2(1 + e) - 1] - 2b^2e \leq b^2(1 + e) - 1 < 0 \tag{29}$$

Both factors of (16) are negative; hence, for all conic section orbits,

$$d\lambda_{\text{abs min}}/dx_f > 0$$

Considering this result, it is possible to state the following conclusions:

1) For transfer between a circular orbit and an external, coplanar, nonintersecting orbit, an absolute minimum-energy two-impulse transfer exists only if $e < 1$, i.e., for those outer orbits that possess a finite maximum distance. In this case, the optimum transfer is the Hohmann ellipse between the circular orbit and the apocentron of the outer orbit, as demonstrated by Horner.[2]

2) For any given $e \geq 1$, where the outer trajectory has no apocentron, *no absolute minimum-energy two-impulse transfer exists*, but there does exist a family of transfers for which λ decreases monotonically as r_2 increases. This family consists of those trajectories that give minimum-energy transfer between the circular orbit and fixed points on the outer orbit.

The discussion in Ref. 1 headed "Transfer Orbit Characteristics" can be extended to cover parabolic outer orbits without modification, except that in the final sentence Eq. (28), of course, should be Eq. (23).

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* Instructor of Aerospace Engineering. Member AIAA.

† Equations (1–24) are in Ref. 1.